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# Chelomei's problem of the stabilization of a statically unstable rod by means of a vibration☆

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ABSTRACT

Chelomei's problem of the stabilization of an elastic, statically unstable rod by means of a vibration is considered. Formulae for the upper and lower critical frequencies for the stabilization of the rod are obtained and analysed. It is shown that, unlike the high-frequency stabilization of an inverted pendulum with a vibrating suspension point, a rod is stabilized by frequencies of a periodic force of the order of the fundamental frequency of the transverse oscillations of the uncompressed rod lying in a certain range.



Chelomei[1–3] pointed out the possibility of increasing the stability of elastic systems by means of vibrations. In particular, he came to the conclusion that an elastic rod, compressed by a longitudinal periodic force with a constant component which exceeds a critical (Eulerian) value, can be stabilized by a high-frequency longitudinal vibration applied to the end of the rod. These results,[1] with minor corrections, appeared in the well-known book by Bogoliubov and Mitropolski[4] and have been discussed by Panovko and Gubanova.[5] In the handbook,[6] Bolotin also analysed the possibility of the stabilization of an unstable elastic rod by means of a longitudinal vibration and, on the basis of numerical calculations, he concluded that there was no analogy with the problem of the stabilization of an inverted pendulum in view of the existence of intermediate resonance zones of instability from high harmonics which constrict the domain of stabilization of the rod.

Chelomei[1,2] writes about the "high frequency" stabilization of a statically unstable rod under the action of a periodic longitudinal force, but does not report the actual values of the frequency and amplitude of the stabilization attained during the course of the experiment. New attempts to investigate the stabilization of a statically unstable rod by means of a high-frequency longitudinal vibration, which were undertaken in connection with this, led to ambiguous conclusions[7,8]. In particular, it was stated[7] that, under the action of a high-frequency vibration, a straight stable equilibrium position exists together with a stable configuration of the bent rod. The effect of an increase in stiffness (in the characteristic frequencies of the transverse oscillations) when a high-frequency longitudinal vibration is applied has been confirmed experimentally[8] but the critical frequency and force corresponding to the loss of stability have not been found.

## 1. Formulation of the problem

Following Chelomei,[1] we consider a thin straight elastic rod of constant cross-section, to the end of which a longitudinal force $P(t) = P_0 + P_t\phi(\omega t)$, periodic with time $t$, is applied. The equation of the transverse oscillations of the rod has the form

$$EJ\frac{\partial^4 u}{\partial x^4} + P(t)\frac{\partial^2 u}{\partial x^2} + 2\gamma m\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0 \qquad (1.1)$$

Here $x$ is the coordinate along the axis of the rod, $u(x, t)$ is the deflection of the rod, $m$ is the mass per unit length, $EJ$ is the flexural stiffness, $\gamma$ is the damping factor, and $P_t$ and $\omega$ are the amplitude and frequency of excitation of the longitudinal vibration. The case when both ends of the rod are simply supported is considered.

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A solution of Eq. (1.1) is sought in the form of a series in the eigenfunctions

$$u(x, t) = \sum_j \varphi_j(t)\sin(j\pi x/l)$$

We substitute this series into Eq. (1.1), multiply by $\sin(k\pi x/l)$, and then integrate over the interval $[0, l]$. As a result, we obtain equations in the functions $\varphi_k(t)$ which, on introducing the new variable $\tau = \omega t$, we write in the form

$$\frac{d^2\varphi_k}{d\tau^2} + 2\beta_k\left(\frac{\Omega_k}{\omega}\right)\frac{d\varphi_k}{d\tau} + \left(\frac{\Omega_k}{\omega}\right)^2\left\{1 - \frac{P_0}{P_k} - \frac{P_t\phi(\tau)}{P_k}\right\}\varphi_k = 0, \quad k = 1, 2, \ldots \tag{1.2}$$

where $\beta_k = \gamma/\Omega_k$, $\Omega_k = \pi^2k^2\sqrt{EJ/m/l^4}$ is the $k$-th normal mode of transverse oscillations of the uncompressed rod and $P_k = \pi^2k^2EJ/l^2$ is the $k$-th critical (Eulerian) force.

The trivial equilibrium position of the rod $u(x, \tau) = 0$ is asymptotically stable, if the function $\varphi_k(\tau) \to 0$ when $\tau \to \infty (k = 1,2,\ldots)$ and unstable if just one of the functions $\varphi_k(\tau)$ becomes unbounded when $\tau \to \infty$.

Chelomei[1] posed the problem of stabiliazing the straight form of an elastic rod by means of longitudinal vibration for a magnitude of the force which exceeds the critical (Eulerian) value $P_0 > P_1$ (that is, of a statically unstable rod). Assuming that the excitation frequency is high compared with the fundamental frequency of the transverse oscillations of the uncompressed rod $\omega \gg \Omega_1$ and applying the perturbation method and the method of averaging to Eq. (1.2) when $k = 1$, he obtained an inequality which describes the domain of stabilization of the rod. In the case when $\phi(\tau) = \cos\tau$, this equality reduces to the form

$$N^2 < \frac{\varepsilon^2}{2\alpha} - 4\beta_1^2; \quad N = \frac{\omega}{\Omega_1}, \quad \varepsilon = \frac{P_t}{P_1}, \quad \alpha = \frac{P_0}{P_1} - 1, \quad \beta_1 = \frac{\gamma}{\Omega_1} \tag{1.3}$$

There is a contradiction here: in deriving this relation it was assumed that $N \gg 1$ but a critical stabilization frequency was of the order of the fundamental frequency. Actually, by putting $\varepsilon = 0.1$, $\alpha = 0.005$, $\beta_1 = 0$, for example, we obtain from inequality (1.3) that $N < 1/\sqrt{10}$. It follows from inequality (1.3) that taking account of damping only reduces the upper limit of the stabilization frequency. Furthermore, there is no lower constraint on the stabilization frequency and, without such a condition, a paradoxical corollary follows: the rod can be stabilized by means of longitudinal vibration which is as slow as desired! Hence, the "high frequency" stabilization of the rod under the action of a longitudinal force[1,2] appears to be in question.

## 2. Analysis of the stability

To analyse the domain of stabilization of the rod we will use the results of an investigation of the stability domains for Hill's equation with damping.[9,10] Applying these results to Eq. (1.2) when $k = 1$ for the case when the constant component of the longitudinal force only slightly exceeds the critical Eulerian value, that is, when $0 < \alpha = P_0/P_1 - 1 \ll 1$, in the case of a small excitation amplitude $\varepsilon = P_t/P_1 \ll 1$ and $\phi(\tau) = \cos\tau$, we obtain an inequality which defines the stabilization domain

$$N^2 < \frac{\varepsilon^2}{2\alpha} - 4\beta_1^2 - \frac{7\alpha}{8} \tag{2.1}$$

It only differs from Chelomei's formula By the last small term.

Moreover, the existence of a lower limit for the frequency $\omega$ follows from the Ince-strutt diagram for the Mathieu–Hill equation.[5,11] In order to obtain the formula describing this limit, it is necessary to analyse the domain of stability arising close to the first critical frequency. We will first consider the case when there is no damping $\beta_1 = 0$. Taking into account the terms of the first and second orders of smallness with respect to $\varepsilon$, we obtain

$$N^2 > H; \quad H = \varepsilon - 2\alpha + \tilde{H}, \quad \tilde{H} = \sqrt{(\varepsilon - 2\alpha)^2 + \frac{\varepsilon^2}{2}} \tag{2.2}$$

Formula (2.2) can also be extended to the case of weak damping. Assuming the square of the critical value of the relative frequency to be equal to $H - \Delta$, where $\Delta$ is a small correction, and using the formula for the first stability domain for the Mathieu–Hill equations. with weak damping,[12] we find

$$\Delta = \frac{2H^2\beta_1^2}{\varepsilon\tilde{H}} \tag{2.3}$$

Finally, by combining relations (2.1)–(2.3), we obtain the domain of stabilization of the rod in the form of a bilateral constraint on the excitation frequency

$$\varepsilon - 2\alpha + \sqrt{(\varepsilon - 2\alpha)^2 + \frac{\varepsilon^2}{2}} - \Delta < N^2 < \frac{\varepsilon^2}{2\alpha} - \frac{7\alpha}{8} - 4\beta_1^2 \tag{2.4}$$

Hence, the boundaries of the stabilization domain depend on three small parameters: $\varepsilon$, $\alpha$ and $\beta_1$. It follows from inequality (2.4) that damping reduces both the upper as well as the lower limit of the stabilization frequency. Note that a stabilization domain only exists in the case of a positive right-hand side of inequality (2.4), that is, for a sufficiently large excitation amplitude.

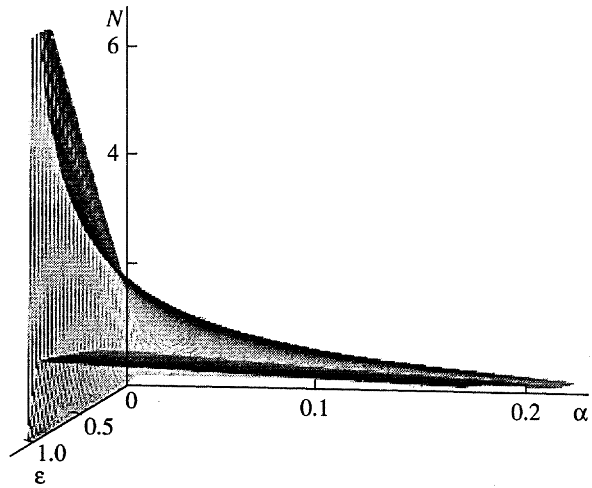


**Fig. 1.**

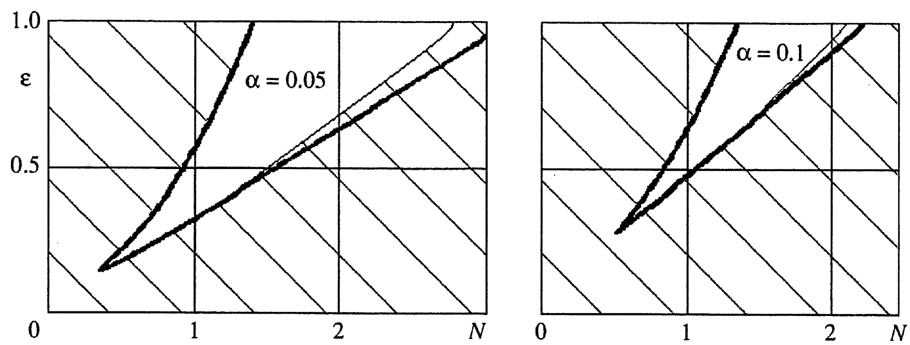


**Fig. 2.**

The dependence of the lower and upper limits of the stabilization frequency on the parameters $\varepsilon$ and $\alpha$ according to formula (2.4) is shown in Fig. 1 for a value of the damping factor $\beta_1 = 0.05$. The stabilization domain is located between two surfaces. The two-dimensional domains of stability of the trivial equilibrium position of the rod under vibration, found when $\beta_1 = 0.05$ and for values of $\alpha = 0.05$ and $\alpha = 0.1$, are shown in Fig. 2. The domains instability (the hatched domains) were obtained numerically using the definition of the monodromy matrix with subsequent calculation of the multipliers of the system and estimates of their moduli (Floquet's method). The bold lines represent the analytical dependences of the frequency on the excitation amplitude according to inequality (2.4). There is clearly good agreement between the analytical and numerical results.

## 3. The effect of higher harmonics on the stabilization domain

We will now estimate the effect of the instability domains (domains of parametric resonance) for Eq. (1.2) when $k = 2, 3, \ldots$ on the stabilization domain found above. In principle, it is possible that the instability domains for Eq. (1.2) when $k = 2, 3, \ldots$ will intersect with the stabilization domain which has been found, thereby constricting the domain of stabilization of the rod. It is well known[9,11] that parametric resonance for the Mathieu–Hill Eq. (1.2) occurs in the case of small amplitudes of the exciting force at frequencies which are close to the values determined by the following equalities

$$\left(\frac{\Omega_k}{\omega}\right)^2\left(1 - \frac{P_0}{P_k}\right) = \frac{n^2}{4}, \quad n = 1, 2, \ldots \tag{3.1}$$

When account is taken of relations (1.2), we obtain from this the values of the critical relative excitation frequencies

$$N = \frac{2k^2}{n}\sqrt{1 - \frac{P_0}{P_1 k^2}}, \quad n = 1, 2, \ldots \tag{3.2}$$

Allowing for the fact that, in the problem in question, the magnitude of $P_0$ is close to the value $P_1$, we write out the first four relative resonance frequencies. We obtain their approximate values from expression (3.2)

$$4\sqrt{3}, 2\sqrt{3}, 4\sqrt{3}/3, \sqrt{3}, \ldots \text{ when } k = 2$$

$$12\sqrt{2}, 6\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}, \ldots \text{ when } k = 3$$

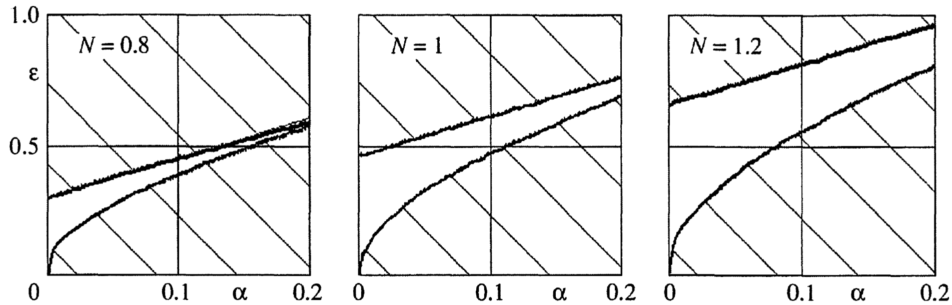


**Fig. 3.**

It is well known that only the first instability domains adjacent to the frequencies $4\sqrt{3\Omega_1}$ and $12\sqrt{2\Omega_1}$ are wide for the Mathieu–Hill Eq. (1.2) and, for moderate excitation amplitudes, the instability corresponding to large values of $n$ disappears when there is weak damping. It follows from this that, when there is damping, the instability domains for Eq. (1.2) for $k = 2, 3, \ldots$ have no effect on the stabilization domain (2.4). Numerical calculations confirm this conclusion. However, this conclusion contradicts results[6] according to which a domain of stabilization of the rod alternates with instability zones from higher harmonics. The stabilization domain found numerically[6] solely for small $\varepsilon$ and $\omega$ agrees qualitatively with the results shown in Fig. 2 for $\alpha = 0.05$.

## 4. Stabilization of a rod with a specified frequency

We will investigate the possibility of stabilization of a rod with a specified longitudinal oscillation frequency assuming that the quantities $\varepsilon$ and $\alpha$ are variable. The corresponding analytical relations can be obtained from bilateral inequality (2.4). In particular, in the case of small $\varepsilon$ and $\beta_1$, the approximate formula for the stabilization domain

$$\alpha < \frac{\varepsilon^2}{2N^2} \tag{4.1}$$

follows from the right-hand side of inequality (2.4).

The stabilization domains corresponding to inequality (2.4) and found numerically (the unhatched regions), are shown in Fig. 3 for $N = 0.8$, 1, 1.2 and $\beta_1 = 0.005$. It follows from Fig. 3 and formula (4.1) that, in the case of a moderate excitation amplitude $\varepsilon$, stabilization of the rod can only be observed for values of the force which only slightly exceed the critical value $\alpha = P_0/P_1 - \ll 1$.

## 5. Conclusion

It has been established from the stability analysis of the solutions of the Mathieu–Hill equation with damping that an elastic rod is stabilized by vibration frequencies of the order of the fundamental frequency of the transverse oscillations lying in a certain range. The analogy between the problem of the stabilization of a rod and the stabilization of an inverted pendulum with a vertically vibrating suspension point, remarked on by Chelomei,[1] is completely natural. In fact, statically unstable systems are stabilized by means of vibration in both systems, and both problems reduce to an analysis of the Ince-steutt diagram (with damping) in the case of negative frequencies close to zero. The difference lies in the fact that, for small excitation amplitudes, the pendulum is stabilized in the upper vertical position by a vibration of the suspension point with frequency which is higher than the natural vibration frequency of the pendulum, while an elastic rod is stabilized by longitudinal vibration frequencies of the order of the fundamental frequency of the transverse oscillations of the uncompressed rod.

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